Available online at www.sciencedirect.com

ScienceDirect

European Journal of Mechanics B/Fluids 26 (2007) 431–443

Electroviscous potential flow in nonlinear analysis of capillary instability

Abd Elmonem Khalil Elcoot

Department of Mathematics, Faculty of Science, El-Faiyum University, El-Faiyum, 63514, Egypt

Received 7 April 2006; received in revised form 15 September 2006; accepted 25 September 2006

Available online 9 November 2006

Abstract

We study the nonlinear stability of electrohydrodynamic of a cylindrical interface separating two conducting fluids of circular cross section in the absence of gravity using electroviscous potential flow analysis. The analysis leads to an explicit nonlinear dispersion relation in which the effects of surface tension, viscosity and electricity on the normal stress are not neglected, but the effect of shear stresses is neglected. Formulas for the growth rates and neutral stability curve are given in general. In the nonlinear theory, it is shown that the evolution of the amplitude is governed by a Ginzburg–Landau equation. When the viscosities are neglected, the cubic nonlinear Schrödinger equation is obtained. Further, it is shown that, near the marginal state, a nonlinear diffusion equation is obtained in the presence of viscosities. The various stability criteria are discussed both analytically and numerically and stability diagrams are obtained. It is also shown that, the viscosity has effect on the nonlinear stability criterion of the system, contrary to previous belief.

1. Introduction

The motion of a liquid jet under the combined influence of capillary force, viscous forces and electric field has received much attention in fluid mechanics literature. Such liquid jet occur in many industrial applications, such as in ink jet printers [1], paint spraying [2], fuel atomization, electrohydrodynamic mixing [3], etc. Most of the analytical solutions stem from the pioneering works of Plateau [4], Rayleigh [5], Weber [6] and Tomotika [7]. The dynamical theory of instability of a long cylindrical column of an inviscid liquid under the action of capillary force was given by Rayleigh [5] following earlier work by Plateau [4] who showed that a long cylinder of liquid is unstable to disturbances with wavelengths greater than the circumference of the cylinder, and those with wavelengths less than the circumference will decay, i.e. the wavenumber k of the symmetrical disturbances are unstable for $k < 1$ and stable for $k > 1$ as the length variable has been nondimensionalized by the undisturbed radius of the cylinder. Rayleigh [5] showed that the maximum growth rate occurs at $k = \pi/4.51$. Rayleigh [8] also considered the capillary instability of a cylindrical liquid viscous jet and neglecting the effect of the outside fluid. The effect of viscosity is treated in the special case in which the viscosity is so great that inertia may be neglected. He shows that the wavelength for maximum growth is very large, strictly infinite. Weber [6] extended Rayleigh [8] by considering an effect of viscosity

E-mail address: aelcoot@hotmail.com (A.E.K. Elcoot).

0997-7546/$ – see front matter
doi:10.1016/j.euromechflu.2006.09.003

and included the action of an inviscid atmosphere of surrounding air on the stability of a columnar jet. He showed that viscosity does not alter the value of the cutoff wavenumber predicted by the inviscid theory and that the influence of the ambient air is not significant if the forward speed of the jet is small. Tomotika [7] extended the solution to include the effect of a viscous surrounding. Tomotika's problem was studied by Lee and Flumerfelt [9] without making the approximations used by Tomotika [7]. They investigated effects of viscosity ratio for various values of Ohnesorge number and a fixed value of the density ratio for some limiting cases. Funada and Joseph [10] treat the general fully viscous problem considered by Tomotika. Theories based on viscous and inviscid potential flows are constructed and compared with the fully analysis and with each other. They found that the growth rates for the inviscid fluid are largest, the growth rates for fully viscous problem are smallest and those of viscous potential flow are between. Funada and Joseph [11] extended the analysis of viscous potential flow [10] to viscoelastic fluids of Maxwell type. Joseph et al. [12] have analyzed Rayleigh–Taylor instability of an Oldroyd fluid using viscoelastic potential flow. They showed that the growth rates for the most unstable wave are much larger than for the comparable viscous drop, which agrees with the surprising fact that the breakup times for viscoelastic drops are shorter. They also do an analysis of Rayleigh–Taylor instability based on viscoelastic potential flow, which gives rise to nearly the same dispersion relation as the unapproximated analysis.

The basic principles for dealing with electrified fluids were developed by Taylor [13–15]. Taylor discovered that it is impossible to account for most electrical phenomena involving moving fluids under the seemingly reasonable assumptions that the fluid is either perfect dielectrics or perfect conductor. The reason is that any perfect dielectric still contains a nonzero free charge density. Although this charge density might be small enough to ignore bulk conduction effects, the charge will typically live on interfaces between fluids. If there is also a nonzero electric field tangent to the interface then there will be a nonzero tangential stress on the interface. The only possible force that can balance a tangential stress is viscous; hence, under these conditions the fluid will necessarily be in motion. This idea has become known as the "leaky dielectric model" for electrically driven fluids. In the light of linear theory of leaky dielectric theory, Melcher and Taylor [16] explained certain paradoxical phenomena pertaining to nonconducting fluids. Its consequences have been successfully compared to experiments on neutrally buoyant drops elongated by electric fields. Saville [17] examined the linear electrohydrodynamic stability of an infinite fluid cylinder in the presence of an axial electric field. Both fluids were treated as leaky dielectrics. He showed that a leaky dielectric requires much lower field strength than a perfect dielectric for jet stabilization to take place. In addition he showed that the stability of the cylindrical configuration depends on the relative magnitude of the conductivity and dielectric constant ratios. A detailed discussion and derivation of the assumptions behind the model are described in the review by Saville [18]. Mestel [19,20] has included surface charge in special limits. Based on a systematic linear computational analysis of the leaky dielectric model, Feng and Scott [21] found that the relationship between the drop deformation parameter and the square of the dimensionless electric field strength is typically nonlinear whereas the linearized asymptotic theory provides merely straight asymptotic lines describing drops with vanishingly small a sphericity at relatively low field strength. They improved agreement between theory and experiment for higher field strengths and larger deformations. Feng [22] extended the computations of Feng and Scott [21] to include the charge convection effect that is expected to emerge when the flow intensity is considerable. In a series of papers [23–27], They developed a nonlinear theory for the stability of a finitely conducting jets of electrically force in different situations. They obtained nonlinear differential equations, Such as Schrödinger equations and Klein–Gordon equations, as well as Ginzburg–Landau equation describing the evolution of the finite amplitude wave packet on a fluid surface with the aid of multiple scales method. They showed that the wavetrain solution of constant amplitude is unstable against modulation if the product of the group velocity rate and the nonlinear interaction parameter is positive. The nonlinear aspects of the model provide new instability regions in the parameter space. They showed that, the nonlinear effects are important in the treatment of electrohydrodynamic phenomena, because the phenomenon cannot be discussed entirely by a linear theory. They found that the uniform axial electric field has a stabilizing or destabilizing influence, according to some of conditions on conductivity and permittivity ratios, for axisymmetric case, was reported by [23]. Elhefnawy et al. [24] found that, the nonlinear stability of the system depends on both the outer and the inner radial electric fields, while, the linear stability depends on the outer field and doesn't depends on the inner one, such effects can only be understood by nonlinear analysis, as the linear analysis fails to predict them. They showed that the streaming has a destabilizing effect, for some conditions on conductivity and permittivity ratios are stated by [25]. They found, the nonlinear stability criterion of interfacial surface charges depends on the outer and the inner dielectric constants, in contrast with the linear analysis, have presented by [26]. Elcoot and Moatimid [27] investigated the nonlinear streaming instability of surface waves propagating through porous media of a cylindrical flow of two concentric fluids when the conductivities of the fluids are finite. The dominant instability strongly depends on the fluid parameters of the jet (viscosity, dielectric constant, finite conductivity) and also the velocity of the fluids. El-Sayed et al. [28,29] studied theoretical the effect of time dependent radial and axial electric fields enhanced heat and mass transfer on the interfacial instability of two conducting fluids separated by a cylindrical interface, and confined between two coaxial circular cylinders. Fing and Bear [30] studied the case of weak viscous effects. This weakness is regarded such that viscous effects appear at the interface and gradually decrease to be ineligible at the bulk. Their treatment based on the viscous contribution has been demonstrated through the normal stress boundary condition. While, the tangential stress is ignored.

In this paper we employ the nonlinear leaky dielectric theory to describe the stability of an electroviscous potential flow analysis for liquid jet. The outline of this paper is as follows. In Section 2, we give description of the problem including the basic equations governing the motion of the considered system together with the appropriate nonlinear boundary conditions. In Section 3, we obtain a nonlinear equation in terms of the interfacial displacement, from which we derive the linear solution (dispersion relation), and the stability conditions for the system are obtained. Also, the breakup phenomena of liquid jets into drops are discuss in the light of linear theory. Section 4, we use the analysis based on the perturbations method and the Fourier transform to derive the Ginzburg–Landau equation describing the evolution of wave packets. For some special cases we derived the cubic nonlinear Schrödinger equation and the nonlinear modified diffusion equation. Also, we compare the linear and the nonlinear theory. In the light of the linear theory, the viscosities μ_1 and μ_2 have no effect on the stability criterion (3.8). This result obtained by Funada and Joseph [10,11] in the absence of electric field for the hydrodynamic jet. But, in the light of the nonlinear theory the modulational stability condition (4.23) depend on the viscosities μ_1 and μ_2. Numerical results of physical parameters of the problem are discussed and stability diagrams are obtained. These show that the electroviscous potential flow analysis is better understood via nonlinear approach. The final Section 5 is conclusion.

2. Problem description

Consider now two incompressible, viscous fluid cylinders, confined between two concentric circular rigid cylinders in a uniform axial electric field of strength E_0. The undisturbed cylindrical interface is taken at radius R. In what follows, the subscripts 1 and 2 denotes variables associated with the fluids inside and outside the interface, respectively. The finite radius of the inner fluid is denoted as R_1, the electrical conductivity by σ_1, the permittivity by ε_1, viscosity by μ_1 and density by ρ_1. Corresponding properties of the outer fluid are R_2, σ_2, ε_2, μ_2 and ρ_2. On the surface between the two fluids, there exists the surface tension T. Both fluids are assumed to be incompressible, and the motion in each fluid is assumed to be irrotational. We shall use a cylindrical system of coordinates (r, θ, z). So that the z-axis is the axis of symmetry of the system in the equilibrium state. We describe the interface profile by the relation $r = R + \eta$, where $\eta = \eta(\theta, z, t)$ is the perturbation in the radius of the interface from its equilibrium value R, and t represents time. While the two solid cores $r = R_1$ and $r = R_2$ are taken as rigids, where $R_1 < R < R_2$.

Here, using the viscous potential flow analysis of capillary jet instability, in which the effects shear stress at the solid cylinders and across the interface between the two fluids are neglected [31]. We consider nonaxisymmetric motion of disturbances, for which the velocity potential $\Psi = \Psi(r, \theta, z, t)$ gives $\mathbf{v} = \nabla\Psi$.

The velocity potential is subject to the equation of continuity:

$$\nabla \cdot \mathbf{v} = 0 \to \nabla^2\Psi = 0; \tag{2.1}$$

thus the potentials for the respective fluids are given by

$$\nabla^2\Psi^{(j)} = 0, \quad j = 1, 2, \tag{2.2}$$

$$\nabla^2\Psi^{(1)} = 0 \quad \text{in } R_1 < r < R + \eta, \tag{2.3}$$

$$\nabla^2\Psi^{(2)} = 0 \quad \text{in } R + \eta < r < R_2, \tag{2.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{2.5}$$

In electrodynamics, it is generally assumed that the quasi-static approximation is valid, i.e., the effects of magnetic induction are negligible which requires that the electric field to be curl free and thus representable as the gradient of electric scalar potentials $\Phi^{(j)}(r,\theta,z,t)$, $j=1,2$. The electric field in both fluids is thus $\underline{E}^{(j)}=E_0\,\underline{e}_z-\nabla\Phi^{(j)}$, where $\underline{e}_z$ is the unit vector along the z-direction. The free charge density is initially zero everywhere within each fluid, it will remains zero. Gauss's law requires the electric scalar potentials also satisfies Laplace's equation:

$$\nabla^2\Phi^{(j)}=0,\quad j=1,2. \tag{2.6}$$

The free surface of location of the interface is defined by the function F

$$F(r,\theta,z,t)=r-R-\eta(\theta,z,t), \tag{2.7}$$

where $F=0$ represents the bounding surface and its normal unit normal vector $\mathbf{n}$ is given by

$$\mathbf{n}=\frac{\nabla F}{|\nabla F|}=\left(1,-\frac{1}{r}\frac{\partial\eta}{\partial\theta},-\frac{\partial\eta}{\partial z}\right)\left[1+\frac{1}{r^2}\left(\frac{\partial\eta}{\partial\theta}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2\right]^{-1/2}. \tag{2.8}$$

The solution for $\Psi^{(j)}$ and $\Phi^{(j)}$ have to satisfy the following boundary conditions:

(i) Boundary conditions at the rigid cylindrical surface $r=R_1$ and $r=R_2$ gives

$$\frac{\partial\Psi^{(j)}}{\partial r}=0,\ \text{at}\ r=R_j, \tag{2.9}$$

$$\frac{\partial\Phi^{(j)}}{\partial z}=0,\ \text{at}\ r=R_j. \tag{2.10}$$

(ii) Boundary conditions at the interface $r=R+\eta$ are given by

The material derivative of F must vanish, yielding the kinematic conditions on the free surface $r=R+\eta$:

$$\frac{DF}{Dt}=\frac{\partial\eta}{\partial t}-\frac{\partial\Psi^{(j)}}{\partial r}+\frac{1}{R^2}\left(1+\frac{\eta}{R}\right)^{-2}\frac{\partial\eta}{\partial\theta}\frac{\partial\Psi^{(j)}}{\partial\theta}+\frac{\partial\eta}{\partial z}\frac{\partial\Psi^{(j)}}{\partial z}=0. \tag{2.11}$$

The tangential component of the electric field is continuous at the interface [32], it is mathematically requires that

$$\big|[E_t]\big|=0, \tag{2.12}$$

where $E_t(=\mathbf{n}\wedge\mathbf{E})$ is the tangential component of the electric field and $|[X]|$ represents the jump across the interface, it is defined as $|[X]|=X_1-X_2$.

The discontinuity in the normal current to the interface; charge accumulation within a material surface element is balanced by conduction from bulk fluid on either side of the surface. At steady state [33], one gets

$$\big|[\sigma E_n]\big|=0, \tag{2.13}$$

where E_n $(=\mathbf{n}\cdot\mathbf{E})$ is the normal component of the electric field.

The dynamical condition that the normal stress should be continuous across the perturbed interface [19,24]:

$$\left|\left[\rho\frac{\partial\Psi}{\partial t}\right]\right|+\frac{1}{2}\big|\big[\rho(\nabla\Psi)^2\big]\big|+\frac{1}{2}\big|\big[\varepsilon E_n^2\big]\big|-\frac{1}{2}\big|\big[\varepsilon E_t^2\big]\big|+2\big|\big[\mu\mathbf{n}\cdot\big[(\mathbf{n}\cdot\nabla)\mathbf{v}\big]\big]\big|=T\nabla\cdot\mathbf{n}. \tag{2.14}$$

3. Solutions

The mathematical system described in the previous section is nonlinear in general, despite the fact that both electric and flow fields are governed by linear partial differential equations. The nonlinearities in the system arise from the boundary conditions at the deformable interface such as capillarity, the coupling of the surface kinematics to the velocity field, etc. In the present work, the nonlinear free boundary problem is simplified by considering only the nonlinear field solutions for first-order perturbation solutions of interface deformations [34–37].

We are interested in the interfacial response between two cylindrical fluids after a disturbance about the equilibrium configuration. Therefore, the surface deflection is expressed by

$$\eta=A\mathrm{e}^{\mathrm{i}(kz+m\theta-\omega t)}+\text{c.c.} \tag{3.1}$$

Here, A is an arbitrary parameter, which determines the behaviour of the amplitude of the disturbance and i $(=\sqrt{-1})$ is the imaginary number. Through the linear theory, the amplitude is assumed to be constant. On the other hand, along the nonlinear approach, A is treated as a slowly varying function of space and time. In addition, k is the axial wave number, which is assumed to be real and positive, m is the azimuthal wave number, which is assumed to be positive integer, ω is the growth rate and c.c. refers to the complex conjugate of the preceding term. It should be note that the imaginary part of ω indicates a disturbance that either grows with time (instability) or decays with time (stability), depending on whether this imaginary part is positive or negative, respectively. As shown by many foregoing works, e.g. Chandrasekhar [38]. The basic variables are expressed in normal modes (3.1). The analysis of the linear stability as presented by [38] is based on neglecting the nonlinear terms from the equations of motion as well as the boundary conditions. Therefore, a dispersion relation should arise without nonlinear terms. The idea for the weak nonlinear approach is a slight departure from the linearity technique. At this stage, the whole problem will contain the linear dispersion with some additional terms that make a correction of the main solution. The weak nonlinear description given here is based on neglecting the nonlinear terms from the equations of motion and appropriate nonlinear boundary conditions. Therefore, the dispersion relation should be extended to include the nonlinear terms.

Since the boundary conditions (2.11)–(2.14) are prescribed at the interface $r = R + \eta$, we express all the physical quantities involved in terms of Taylor's series about $r = R$. On substituting the solutions of the basic equations (2.2) and (2.6) into the boundary conditions (2.11)–(2.14), we get the scalar electric potential $\Phi^{(j)}$ and the velocity potential $\Psi^{(j)}$ solutions, in terms of the elevation parameter η. Inserting the solutions of $\Phi^{(j)}$ and $\Psi^{(j)}$ into the normal stress tensor (2.14). Keeping in mind that the elevation function η is small, the use of binomial expansion is convenient. The calculations are lengthy but straightforward. Up to the third order of η, one finds the following nonlinear equation in the interfacial displacement η [37]:

$$D(\omega,k,m)\eta = \Omega_1(\omega,k,m)\eta^2 + \Omega_2(\omega,k,m)\eta^3 + \cdots, \tag{3.2}$$

where

$$\begin{aligned} D(\omega,m,k) = {}& \frac{T}{R^2}\bigl(1-m^2-k^2R^2\bigr) + \frac{kE_0^2 g_1(k)g_2(k)(\varepsilon_1-\varepsilon_2)(\sigma_1-\sigma_2)}{d_1(k)} \\ & - \frac{\omega(2\mathrm{i}k^2\mu_1 N_1(k) + \rho_1 M_1(k)(k)\omega)}{\lambda_1(k)} + \frac{\omega(2\mathrm{i}k^2\mu_2 N_2(k) + \rho_2 M_2(k)\omega)}{\lambda_2(k)} = 0, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d_1(k) &= \sigma_1 g_2(k)G_1(k) - \sigma_2 g_1(k)G_2(k), \\ \lambda_j(k) &= k\bigl(I'_m(kR_j)K'_m(kR) - I'_m(kR)K'_m(kR_j)\bigr), \\ g_j(k) &= I_m(kR_j)K_m(kR) - I_m(kR)K_m(kR_j), \\ G_j(k) &= K_m(kR_j)I'_m(kR) - I_m(kR_j)K'_m(kR), \\ M_j(k) &= K_m(kR)I'_m(kR_j) - I_m(kR)K'_m(kR_j), \\ N_j(k) &= K'_m(kR_j)I''_m(kR) - I'_m(kR_j)K''_m(kR), \end{aligned}$$

where the symbols $I_m(kR)$ and $K_m(kR)$ are the modified Bessel functions of order m and the prime on the modified Bessel functions denotes the differentiation with respect to r when $r = R_1$, R_2 or R. The nonlinear terms $\Omega_1(\omega,k,m)$ and $\Omega_2(\omega,k,m)$ in Eq. (3.2) are describe the behaviour of the physical parameters of the system of the second- and third-order terms, respectively. The calculations are lengthy but straightforward, In order to facilitate the tedious computations the MATHEMATICA package has been used. According to the general theory of Grimshaw [34], the nonlinear terms $\Omega_1(\omega,k,m)$ and $\Omega_2(\omega,k,m)$ of the characteristic equation (3.2) consist of two terms. One part contains the interaction of the second harmonic term, together with the cubic interaction of the primary harmonic term, which is identical to the same term that arises in the Stokes expansion for plane progressive wave. The nonlinear terms $\Omega_1(\omega,k,m)$ and $\Omega_2(\omega,k,m)$ are given in Appendix A for spacial case, that we need later.

Eq. (3.3) is reduced to the dispersion relation

$$a_2{\omega_0}^2 + a_1\omega_0 + a_0 = 0, \tag{3.4}$$

where

$$a_2 = \frac{\rho_1 M_1(k)}{\lambda_1(k)} - \frac{\rho_2 M_2(k)}{\lambda_2(k)},$$
$$a_1 = \frac{2k^2\mu_1 N_1(k)}{\lambda_1(k)} - \frac{2k^2\mu_2 N_2(k)}{\lambda_2(k)},$$
$$a_0 = \frac{T}{R^2}\left(k^2R^2 - m^2 - 1\right) + \frac{kE_0^2 g_1(k) g_2(k)(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2)}{d_1(k)},$$

and $\omega_0 = -\mathrm{i}\omega$. Solving (3.4) we get

$$\omega_0 = \frac{1}{2a_2}\left(-a_1 \pm \sqrt{a_1^2 + 4a_0a_2}\right). \tag{3.5}$$

Here, it is easy to find that $\omega_0 = 0$ gives a relation independent of viscosity. In other words, the relation holds even for inviscid fluids; this is helpful for the problem to be considered herein

If we suppose that $R_1 = 0$ and $R_2 \to \infty$, and $E_0 = 0$ the dispersion relation (3.4) becomes

$$a_2^* {\omega_0}^2 + a_1^*\omega_0 + a_0^* = 0, \tag{3.6}$$

with

$$a_2^* = \frac{\rho_1 I_m(kR)}{kI_m'(kR)} - \frac{\rho_2 K_m(kR)}{kK_m'(kR)},$$
$$a_1^* = 2k\mu_2\left(\frac{1}{kR} - \frac{K_m(kR)}{K_m'(kR)}\right) + 2k\mu_1\left(\frac{1}{kR} + \frac{I_m(kR)}{I_m'(kR)}\right),$$
$$a_0^* = \frac{T}{R^2}\left(k^2R^2 + m^2 - 1\right) + \frac{2k(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2)E_0^2}{\sigma_1 I_0'(2kR)K_0(2kR) - \sigma_2 I_0(2kR)K_0'(2kR)}.$$

Nonaxisymmetric deformations ($m \neq 0$) are always stable; axisymmetric deformations ($m = 0$) with wavenumbers within the range $0 < kR < 1$ are unstable [8]. The formula for the axisymmetric case when $E_0 = 0$ was reported by Funada and Joseph [11].

Applying the Routh–Hurwitz criterion to (3.6). we obtain conditions for stability (in other words, to have the real part of ω_0 is less than zero or the imaginary part of ω is also less than zero):

$$a_1^* > 0 \quad \text{and} \quad a_0^* > 0, \tag{3.7}$$

since a_2^* is always positive. From above we notice that the condition $a_1^* > 0$ is trivially satisfied since μ_1 and μ_2 are always positive, while the condition $a_0^* > 0$ reduces to

$$\frac{T}{R^2}\left(k^2R^2 + m^2 - 1\right) + \frac{k(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2)E_0^2}{\sigma_1 I_0'(kR)K_0(kR) - \sigma_2 I_0(kR)K_0'(kR)} > 0. \tag{3.8}$$

It is clear that, in axisymmetric case ($m = 0$) the electric field has a stabilizing influence on the wave motion. This theoretical result was obtained by Elhefnawy et al. [23]. It is clear that the viscosity coefficients (μ_1 and μ_2) has no effect on the stability condition (3.8). This result obtained by Funada and Joseph [11] when $E_0 = 0$.

For values of $E_0 \geqslant E_c$ (E_c is the critical electric field) where

$$E_c^2 = \frac{T(1 - k^2R^2 - m^2)\sigma_1 I_0'(kR)K_0(kR) - \sigma_2 I_0(kR)K_0'(kR)}{kR^2(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2)}, \tag{3.9}$$

the system is linearly stable. For $E_0 < E_c$ the system is unstable. It is found that the liquid jet is stable for all purely nonaxisymmetric deformations, but is unstable for symmetric varicose deformations with wavelengths exceeding the circumference of the cylinder for $(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2) > 0$. It is follows that the most interesting mode of disturbance is the axisymmetric mode. Therefore, from now on, the axisymmetric mode ($m = 0$) is only considered.

Now we shall discuss, the breakup phenomena of liquid jets into drops for the dispersion relation (3.4) in the light of linear theory of the mode $m = 0$. The growth rate curves ω_0 versus k depend only on control parameters $\varepsilon_{1,2}$, $\sigma_{1,2}$, E_0, $\rho_{1,2}$, $\mu_{1,2}$ and T are shown in Figs. 1 and 2. The numerical results inspect the maximum growth rate ω_0 of the instability of the system.

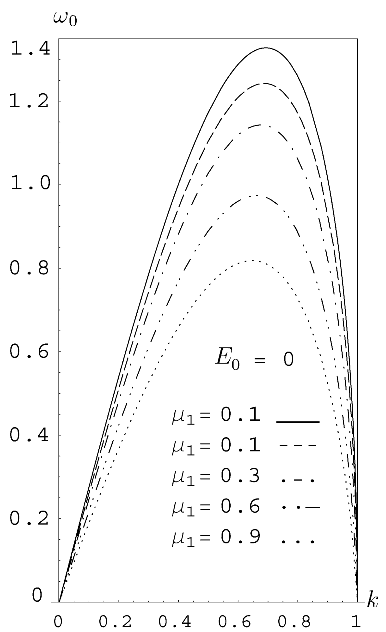

Fig. 1. Stability diagram on the ω_0–k plane, according to Eq. (3.4), for different values of μ_1, and for linear system having $m = 0$, $R = 2$ cm, $R_1 = 1$ cm, $R_2 = 3$ cm, $T = 20$ dyn cm^{-1}, $\mu_2 = 0$, $\rho_1 = 0.99823$ gm cm^{-3}, $\rho_2 = 0.79$ gm cm^{-3}, $\varepsilon_1 = 80.37$, $\varepsilon_2 = 2.5$, $\sigma_1 = 0.2$ s^{-1}, $\sigma_2 = 0.1$ s^{-1} and $E_0 = 0$.

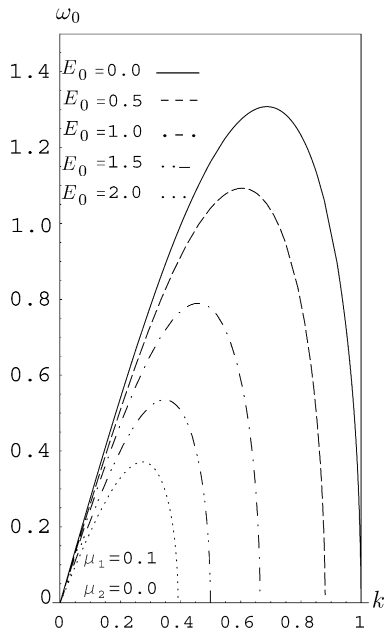

Fig. 2. Stability diagram for the same system as in Fig. 1, but with $\mu_1 = 0.1$ gm cm^{-1} s^{-1} and $\mu_2 = 0$, for different values of E_0 (gm$^{1/2}$ cm$^{-1/2}$ s^{-1}).

In Fig. 1, the dispersion curves are obtained using the calculated values of the growth rate for different wavenumbers and for five μ_1 corresponding to the cases $\mu_2 = 0$ and $E_0 = 0$. As predicted by the linear theory, the viscosity reduces the magnitude of the growth rate for all wavenumbers. In addition, the maximum growth rate occurs at lower wavenumbers for more viscous jets. This is due to the more effective viscous damping at larger wavenumbers. Fig. 2 represents the same system as considered in Fig. 1, but when $\mu_1 = 0.1$ and $\mu_2 = 0$, for some values of E_0. It is clear that, the liquid jet has a linear stabilizing effect. It is found that, the electric field E_0, reduce the breakup phenomena of liquid jets into drops.

4. The axisymmetry self-modulation waves

To investigate the nonlinear stability of the considered system, for the special case, $R_1 = 0$ and $R_2 \to \infty$ when $m = 0$, we may introduce a modulation to the problem so that the linear dispersion relation $D(\omega, k)$ represents a slowly modulated wavetrain. To do this, let us introduce the following perturbation expansion for η [39]:

$$\eta = \delta\eta_1 + \delta^2\eta_2 + \delta^3\eta_3 + \cdots, \tag{4.1}$$

where δ is a small parameter (i.e. $\delta \ll 1$) representing the strength of the nonlinearity, which measures the ratio of a typical wavelength or periodic time relative to a typical length or timescale of modulation (measures the steppes ratio as the perturbation parameter). Here

$$\eta_1 = A(z,t)\exp\bigl[\mathrm{i}(kz - \omega t)\bigr] + \text{c.c.}, \tag{4.2}$$

where η_2 (a second-harmonic term) is given by

$$\eta_2 = \Omega A^2 \exp\bigl[2\mathrm{i}(kz - \omega t)\bigr] + \text{c.c.}, \tag{4.3}$$

and

$$\eta_3 = |A|^2 A \exp\bigl[\mathrm{i}(kz - \omega t)\bigr] + \text{c.c.} \tag{4.4}$$

On substituting expressions (4.1)–(4.4) into Eqs. (3.2), and equating the coefficients of terms of equal powers in δ. We obtain the linear characteristic function

$$D(\omega,k)=\left(\rho_1\frac{I_0(kR)}{kI_0'(kR)}-\rho_2\frac{K_0(kR)}{kK_0'(kR)}\right)\omega^2+2\mathrm{i}\left[\mu_2\left(\frac{1}{kR}-\frac{K_0(kR)}{K_0'(kR)}\right)-\mu_1\left(\frac{1}{kR}+\frac{I_0(kR)}{I_0'(kR)}\right)\right]\omega$$
$$+\frac{T}{R^2}\left(1-k^2R^2\right)-\frac{k(\varepsilon_1-\varepsilon_2)(\sigma_1-\sigma_2)E_0^2}{\sigma_1 I_0'(kR)K_0(kR)-\sigma_2 I_0(kR)K_0'(kR)}=0. \tag{4.5}$$

The problem can be solved for any order with the knowledge of the solutions of all the previous orders, we get

$$\Omega=\frac{\Omega_1}{D(2\omega,2k)}, \tag{4.6}$$

and

$$D\left(\omega,k,|A|^2\right)=D(\omega,k)-\delta^2G|A|^2=0, \tag{4.7}$$

where

$$G=\frac{2\Omega_1^2}{D(2\omega,2k)}+\Omega_2. \tag{4.8}$$

The nonlinear term $G|A|^2$ in Eq. (4.7) arises by self-interaction and the coefficient G is the nonlinear term, the "Landau constant". It is describe the behaviour of the physical parameters of the system in the nonlinear approach. The calculations are lengthy but straightforward, mostly of third order nonlinear, $\Omega_1=\Omega_1(\omega,k,m)$ and $\Omega_2=\Omega_2(\omega,k,m)$ at $m=0$ are given in Appendix A.

To discuss the nonlinear stability problem for nonlinear characteristic function (4.7), we may introduced an modulation to the problem, so that the linear dispersion relation $D(\omega,k)$ represents a slowly modulated wavetrain. To do this, first, we expand $D(\omega,k,|A|^2)$ as a function of ω, k, and $|A|^2$ in a Taylor series about the wavenumber k_0, the angular frequency ω_0, and the constant amplitude A_0. Therefore,

$$D+\frac{\partial D}{\partial\omega}\varpi+\frac{\partial D}{\partial k}K+\frac{1}{2}\left[\frac{\partial^2 D}{\partial\omega^2}\varpi^2+2\frac{\partial^2 D}{\partial\omega\partial k}K\varpi+\frac{\partial^2 D}{\partial k^2}K^2\right]=\delta^2G|A|^2, \tag{4.9}$$

where ϖ and K are small, and $\omega-\omega_0=\varpi$, $k-k_0=K$. The nonlinear coefficient G evaluated at $A=A_0=0$, is given by $G=(\partial D/\partial|A|^2)\,|_{A_0}$. Suppose that

$$\varpi=\sum_{n=1}^{\infty}\delta^n\varpi_n,\qquad K=\sum_{n=1}^{\infty}\delta^nK_n, \tag{4.10}$$

where

$$\varpi_n=\varpi/\delta^n,\qquad K_n=K/\delta^n. \tag{4.11}$$

We can rewrite η in the form

$$\eta=A(z,t)\mathrm{e}^{\mathrm{i}(k_0z-\omega_0t)}+\text{c.c.} \tag{4.12}$$

Now, using the Fourier transform of the envelope function $A(\varpi,K)$ and its inverse transform $A(z,t)$ of the forms

$$A(\varpi,K)=\int_{-\infty}^{\infty}A(z,t)\mathrm{e}^{\mathrm{i}(\varpi t-Kz)}\,\mathrm{d}t\,\mathrm{d}z, \tag{4.13}$$

$$A(z,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}A(\varpi,K)\mathrm{e}^{-\mathrm{i}(\varpi t-Kz)}\,\mathrm{d}\varpi\,\mathrm{d}K. \tag{4.14}$$

At the second and third order problems we obtained the following system of partial differential equations

$$-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial t}+\frac{\partial D}{\partial k}\frac{\partial A}{\partial z}=0, \tag{4.15}$$

$$\mathrm{i}\left(-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial t}+\frac{\partial D}{\partial k}\frac{\partial A}{\partial z}\right)+\frac{1}{2}\frac{\partial^2 D}{\partial \omega^2}\frac{\partial^2 A}{\partial t^2}-\frac{\partial^2 D}{\partial \omega\partial k}\frac{\partial^2 A}{\partial z\partial t}+\frac{1}{2}\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 A}{\partial z^2}=\delta^2 G|A|^2A, \tag{4.16}$$

where the coefficients of the linear terms are simply derivatives of the linear characteristic function (4.5). Combining the two Eqs. (4.15) and (4.16) following similar arguments as given by Elcoot and Moatimid [27], we rewrite Eq. (4.16) to become

$$\mathrm{i}\left(\frac{\partial A}{\partial \mathcal{T}}+\frac{\mathrm{d}\omega}{\mathrm{d}k}\frac{\partial A}{\partial \mathcal{Z}}\right)+\delta(P_r+\mathrm{i}P_i)\frac{\partial^2 A}{\partial \mathcal{Z}^2}=\delta(Q_r+\mathrm{i}Q_i)|A|^2A, \tag{4.17}$$

where $\mathcal{Z}=z\delta$ and $\mathcal{T}=t\delta$ are slow space and time variables, and

$$P_r+\mathrm{i}P_i=\frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}, \tag{4.18}$$

$$Q_r+\mathrm{i}Q_i=\frac{G}{\partial D/\partial\omega}. \tag{4.19}$$

The nonlinear complex coefficients $(\mathrm{d}^2\omega/\mathrm{d}k^2)/2$ and $-G/(\partial D/\partial\omega)$ are the group velocity rate and the nonlinear interaction parameters, respectively. Here, P_r, P_i, Q_r, and Q_i are real values. It is evident that, in the case of a coordinate system moving with the group velocity $\mathrm{d}\omega/\mathrm{d}k$, Eq. (4.17) needs to be rescaled. This can readily be accomplished by introducing the independent variables

$$\xi=\mathcal{Z}-(\mathrm{d}\omega/\mathrm{d}k)\mathcal{T} \quad \text{and} \quad \zeta=\mathcal{T}\delta, \tag{4.20}$$

so that in the $(\mathcal{Z},\mathcal{T})$ coordinates Eq. (4.17) becomes

$$\mathrm{i}\frac{\partial A}{\partial \zeta}+(P_r+\mathrm{i}P_i)\frac{\partial^2 A}{\partial \xi^2}=(Q_r+\mathrm{i}Q_i)|A|^2A. \tag{4.21}$$

Eq. (4.21) is the well-known Ginzburg–Landau equation. It may be used to study the stability behaviour of the considered model. Lange and Newell [40] derived the stability criteria of this equation. If the solution of Eq. (4.21) is linearly perturbed, the perturbations are stable if both the following conditions.

$$P_rQ_r+P_iQ_i>0 \quad \text{and} \quad Q_i<0, \tag{4.22}$$

are satisfied. Otherwise, the system is unstable (i.e. the system does not oscillate about its equilibrium state). Here the real parts P_r and Q_r do not depend on the viscosities μ_1 and μ_2, while the imagery parts P_i and Q_i are function of μ_1 and μ_2. The transition curves separating the stable from the unstable region correspond to

$$P_rQ_r+P_iQ_i=0, \tag{4.23}$$

$$Q_i=0. \tag{4.24}$$

These marginal curves may be born out of numerical estimation. It is worthwhile, to notice that the linear theory have no implication in the stability criterion (3.8) for viscosities μ_1 and μ_2. But, in the nonlinear theory the stability condition (4.23) depend on the viscosities μ_1 and μ_2.

To support the analytical approach for the special case, $R_1=0$ and $R_2\to\infty$ when $m=0$, we shall discuss, the numerical results in the light of linear and nonlinear approach, for physical parameters of the problem. It is convenient for the linear and nonlinear numerical discussion to write the stability conditions in an appropriate dimensionless form. This can be do in a number of ways depending primarily on the choice of the characteristic length. Consider the following dimensionless forms: the characteristic length $=R$, the characteristic time $=1/\omega$ and the characteristic mass $=T/\omega^2$. The other dimensionless quantities are given by

$$\begin{gathered} k=\frac{k^*}{R}, \qquad \rho_j=\rho_j^*\left(T/\omega^2R^3\right)^{1/2}, \qquad \mu_j=\mu_j^*T/\omega R, \\ \sigma_j=\sigma_j^*/\omega, \qquad E_j=E_j^*(T/R)^{1/2}, \end{gathered} \tag{4.25}$$

where the superposed asterisks refer to dimensionless quantities. From now on, it will be omitted for simplicity. To this end, the interface of the system of the dispersion relation (4.5) becomes stable iff

$$\left(k^2-1\right)+\frac{k(\varepsilon_1-\varepsilon_2)(\sigma-1)E_0^2}{\sigma I_0'(k)K_0(k)-I_0(k)K_0'(k)}\geqslant 0, \tag{4.26}$$

where $\sigma=\sigma_1/\sigma_2$. The interface of the system is linearly stable or unstable depending on whether the electric field E_0 is larger or smaller than E_c, where

$$E_c^2=\frac{(1-k^2)(\sigma I_0'(k)K_0(k)-I_0(kR)K_0'(k))}{k(\varepsilon_1-\varepsilon_2)(\sigma-1)}. \tag{4.27}$$

The critical electric field E_c separates stable regions from unstable regions.

In Fig. 3, we shall discuss numerically, the electroviscous potential flow analysis of capillary instability of Ginzburg–Landau equation (4.21).The analysis of the system depends on the linear condition (4.26) and the nonlinear stability conditions (4.22) through the dimensionless form (4.25). The next step in the analysis of viscous potential flow analysis of capillary instability through electric media is to determine the stable and unstable regions through the nonlinear theory, for example, the behaviour of the electric field in the parameter space. The linear stability can be discussed by dividing (E_0^2–σ)-plane into stable S (above the linear transition curve) and unstable region U (below the linear transition curve). The continued line in Figs. 3 and 4, represents the linear transition curve representing the relation (4.26). The nonlinear transition curves (4.23), (4.24) will divide the linear stable region into stable and new nonlinear unstable parts. The nonlinear curves produce newly unstable regions U_1, U_2, which were stable in the sense. The comparison between the linear and nonlinear theory make the stability of system more accurately. Through Fig. 3 the transition curve $P_rQ_r+P_iQ_i=0$ may be rearranged in a fifth-degree polynomial on E_0^2, while the transition curve $Q_i=0$ may be rearranged in a third-degree polynomial on E_0^2. Some roots of these nonlinear curves (branches) do not appear in Fig. 3, because it lie in the negative part of E_0^2-axis. The dotted lines indicate the nonlinear curve

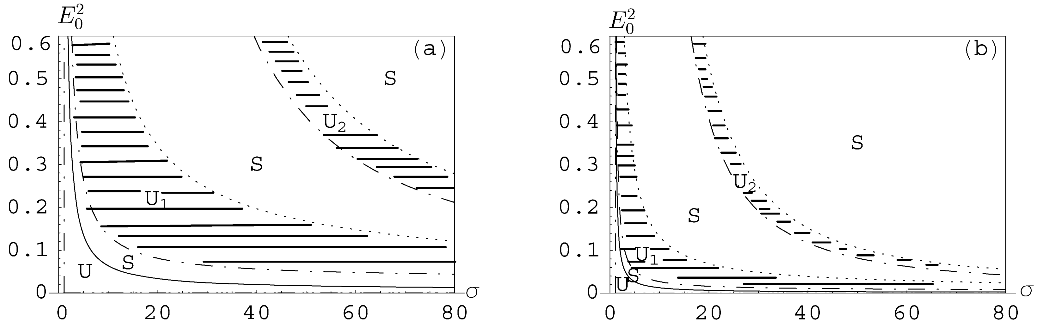

Fig. 3. Stability diagram in view of the nonlinear stability approach according to relations (4.23), (4.24) and (4.27) for system having $\rho_1=0.02$, $\rho_2=0.01$, $\varepsilon_1=80.37$, $\varepsilon_2=2.5$, $\mu_1=0.1$, $\mu_2=0$, where (a) refers to the case $k=0.1$, (b) refers to the case $k=0.3$. The continued line represents the linear curve (4.27), the dashed lines represents (4.23) and the dashed dotted lines represents (4.24).

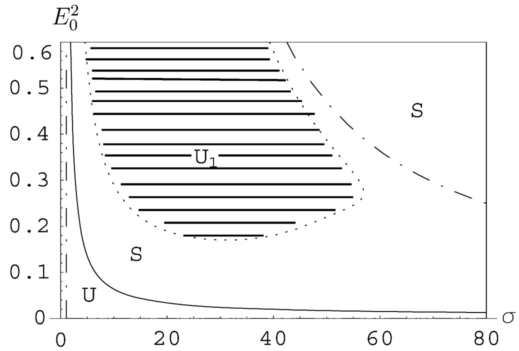

Fig. 4. Stability diagram for the system as considered in Fig. 3, but with $\mu_1=2$.

$P_r Q_r + P_i Q_i = 0$ and the dashed dotted lines represent the nonlinear curve $Q_i = 0$. Fig. 3, displays the stability diagram in the (E_0^2–σ)-plane according to the cases, $k = 0.1$ (Fig. 3(a)) and $k = 0.3$ (Fig. 1(b)). Comparing Figs. 3(a) and 3(b), we observe that the linear curve shift downwards, i.e., the linear stable region S increases with the increase of k. This means that, the stabilizing influence of the field, in the linear theory with respect to the velocity k. The nonlinear curve $P_r Q_r + P_i Q_i = 0$ has two branches as in Fig. 3. It is also observed that the newly unstable regions U_1 is confined between the lower branches of $Q_i = 0$ and $P_r Q_r + P_i Q_i = 0$. While unstable regions U_2 appears between the upper branches of them. Through the nonlinear approach, It is clear that, the new unstable regions U_1 and U_2 decrease as k increases which means that the electric field has stabilizing effect in the light of nonlinear theory with increasing k.

Fig. 4 represents the same system considered in Fig. 3(a), but when $\mu_1 = 2$. We observe that the unstable region U_1 appears in Fig. 4 between the two nonlinear branches $P_r Q_r + P_i Q_i = 0$. While the other region U_2 disappears. We also observe that the linear curve in Fig. 3(a) is the same as in Fig. 4, which means that the linear curve does not depend on μ_1. From the numerical discussion, it is evident that, the parameter μ_1 has a destabilizing influence. This is in contrast to the linear theory. It seems in general that electroviscous potential flow analysis is a nonlinear phenomenon and it is better understood via nonlinear approach.

As a special case, if one considers that the interfacial viscosities are neglected, i.e., $\mu_1 = 0$ and $\mu_2 = 0$. Therefore, in (4.21), both P_i and Q_i equal zero. In this case, (4.21) reduced to the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial A}{\partial \zeta} + P_r \frac{\partial^2 A}{\partial \xi^2} = Q_r |A|^2 A. \tag{4.28}$$

The solutions of the above (4.28) are stable if [23,41] $P_r Q_r > 0$. In addition, the solution of (4.28) is linearly perturbed, the perturbations are stable if condition $P_r Q_r > 0$ is satisfied. On the other hand, if $P_r Q_r < 0$ the system is unstable (i.e., the system does not oscillate about the steady state) and solitary waves propagate though the interface. The solution of (4.28) also breaks down as denominator of Ω tends to zero. Physically, it represents the second harmonic resonance. The solution of Eq. (4.28) is discussed in detail in [23].

When the effect of the viscosities μ_1 and μ_2 are considered, i.e., $\mu_1 \neq 0$ and $\mu_2 \neq 0$, for the special case, $R_1 = 0$ and $R_2 \to \infty$ when $m = 0$. Since the linear dispersion relation (3.6) is quadratic in $(-\mathrm{i}\omega)$, the principle of overstability is valid and we study (4.21) near the marginal state. Therefore, $P_r = Q_r = 0$, and (4.21) reduced to the nonlinear diffusion equation

$$\frac{\partial A}{\partial \zeta} + P_i \frac{\partial^2 A}{\partial \xi^2} = Q_i |A|^2 A. \tag{4.29}$$

It is known that the solutions of (4.29) are bounded if $P_i < 0$ and $Q_i < 0$. An equation similar to (4.29) was derived for waves in cylinder was by Iizuka and Wadati [42] and Elhefnawy [43].

5. Conclusion

The instability of a cylindrical interface separating two, viscous, conducting and incompressible fluids in the presence of an axial electric field is studied with allowance for small, but finite, disturbances and for spatial as well as temporal development. Using electroviscous potential flow analysis. It satisfy the Navier–Stokes equations. Because the no-slip condition cannot be satisfied the effects of shear stresses are neglected, but the effects of extensional stresses at the interface on the normal stresses are fully represented. By using nonlinear approach, we obtain a dispersion relation in the linear approximation and a generalized formulation of the amplitude equation in the nonlinear approximation. The results in the linear problem show that the viscosities no effect, while the electric field has a stabilizing effect.

In the nonlinear analysis, we obtain Ginzburg–Landau equation. When viscosities are neglected, the nonlinear Schrödinger equation is obtained. In the presence of viscosities the nonlinear diffusion equation is obtained near the marginal state. From the numerical discussion it is evident that, the viscosity plays an important role in the nonlinear stability criterion of the problem. It is found that the viscosity plays a dual role in the stability criterion in contrast with the linear analysis [10].

Appendix A

The nonlinear terms Ω_1 and Ω_2 of (3.2) for the special case, $R_1 = 0$, $R_2 \to \infty$ and $m = 0$, or as in Eq. (4.8) are

$$\begin{aligned}\Omega_1 = {} & -\left(\frac{1}{R^3} + \frac{k^2}{2R}\right)T - \frac{\rho_1}{2}\left[\omega\gamma_2 I_0(kR) + k^2\gamma_1^2\left(I_0^2(kR) - I_0'^2(kR)\right)\right] \\ & + \frac{1}{2}\left[E_0^2\varepsilon_1\left(2\mathrm{i}k\alpha_2 I_0(kR) + k^2\alpha_1^2 I_0^2(kR) + \alpha_1 k^2 I_0'(kR)\left(4\mathrm{i} + \alpha_1 I_0'(kR)\right)\right)\right] \\ & - \frac{\rho_2}{2}\left[\omega\delta_2 K_0(kR) + k^2\delta_1^2\left(K_0^2(kR) - K_0'^2(kR)\right)\right] \\ & - \frac{1}{2}\left[E_0^2\varepsilon_2\left(2\mathrm{i}k\beta_2 K_0(kR) + k^2\beta_1^2 K_0^2(kR) + \beta_1 k^2 K_0'(kR)\left(4\mathrm{i} + \alpha_1 K_0'(kR)\right)\right)\right] \\ & - 2\mathrm{i}\mu_1 k^2\left(2k\alpha_1 K_0'(kR) + \alpha_2 I_0''(kR)\right) + 2\mathrm{i}\mu_2 k^2\left(2k\delta_1 K_0'(kR) + \delta_2 K_0''(kR)\right),\end{aligned}$$

$$\begin{aligned}\Omega_2 = {} & \left(-\frac{3k^4}{2} + \frac{1}{R^4} + \frac{k^2}{2R^2}\right)T - \rho_1\left[\omega\gamma_3 I_0(kR) + k^2\gamma_1\gamma_2\left(I_0^2(kR) - I_0'^2(kR)\right)\right] \\ & + E_0^2\varepsilon_1\left[k^2\alpha_1\alpha_2 I_0^2(kR) + k^2\alpha_2 I_0'(kR)\left(2\mathrm{i}\alpha_3 + \alpha_1 I_0'(kR)\right)kI_0(kR)\left(\mathrm{i}\alpha_3 + 2k^2\alpha_1^2 I_0'(kR)\right)\right] \\ & + \rho_2\left[\omega\delta_3 K_0(kR) + k^2\delta_1\delta_2\left(K_0^2(kR) - K_0'^2(kR)\right)\right] \\ & - E_0^2\varepsilon_1\left[k^2\beta_1\beta_2 K_0^2(kR) + k^2\beta_2 K_0'(kR)\left(2\mathrm{i}\beta_3 + \beta_1 I_0'(kR)\right)kK_0(kR)\left(\mathrm{i}\beta_3 + 2k^2\beta_1^2 K_0'(kR)\right)\right] \\ & + 2\mathrm{i}\mu_1 k^2\left(2k^4\gamma_1 I_0(kR) + k^2\left(\gamma_2 I_0'(kR) + \gamma_3 I_0''(kR) + k^2\gamma_1 I_0''(kR)\right)\right) \\ & - 2\mathrm{i}\mu_2 k^2\left(2k^4\delta_1 K_0(kR) + k^2\left(\delta_2 K_0'(kR) + \delta_3 K_0''(kR) + k^2\delta_1 K_0''(kR)\right)\right),\end{aligned}$$

where

$$\begin{aligned}\gamma_1 &= -\omega/kI_0'(kR), \\ \gamma_2 &= -\omega I_0(kR)/I_0'^2(kR), \\ \gamma_3 &= -k\omega I_0^2(kR)/I_0'^3(kR), \\ \delta_1 &= -\omega/kK_0'(kR), \\ \delta_2 &= -\omega K_0(kR)/K_0'^2(kR), \\ \delta_3 &= -k\omega K_0^2(kR)/K_0'^3(kR), \\ \alpha_1 &= -\mathrm{i}E_0(\sigma_1 - \sigma_2)K_0(kR)/D_1, \\ \alpha_2 &= \frac{k}{D_1^2}\mathrm{i}E_0(\sigma_1 - \sigma_2)\left\{-\sigma_2 K_0(kR)K_0'(kR)I_0'(kR) + I_0(kR)\left(k(\sigma_1 - \sigma_2)K_0^2(kR) + \sigma_2 K_0'^2(kR)\right)\right\}, \\ \alpha_3 &= \frac{k}{D_1^2}\mathrm{i}E_0(\sigma_1 - \sigma_2)\left\{\sigma_2 K_0'(kR)\left(kK_0(kR)I_0(kR) + K_0'(kR)I_0'(kR)\right) + \left\{K_0(kR)\left(kR^3 D_1\left(\sigma_2 K_0(kR)I_0'(kR)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. + \sigma_1 I_0(kR)K_0'(kR)\right) + (\sigma_1 - \sigma_2)^2\left(K_0(kR)I_0(kR) + K_0'(kR)I_0'(kR)\right)^2\right)/D_1\right\}\right\},\end{aligned}$$

$$\begin{aligned}\beta_1 &= -\mathrm{i}E_0(\sigma_1 - \sigma_2)I_0(kR)/D_1, \\ \beta_2 &= \frac{k}{D_1^2}\mathrm{i}E_0(\sigma_1 - \sigma_2)\left\{-\sigma_1 I_0(kR)K_0'(kR)I_0'(kR) + K_0(kR)\left(k(\sigma_1 - \sigma_2)I_0^2(kR) + \sigma_1 K_0(kR)I_0'^2(kR)\right)\right\}, \\ \beta_3 &= \frac{k}{D_1^2}\mathrm{i}E_0(\sigma_1 - \sigma_2)\left\{\sigma_2 I_0'(kR)\left(kK_0(kR)I_0(kR) + K_0'(kR)I_0'(kR)\right) + \left\{I_0(kR)\left(kR^3 D_1\left(\sigma_2 K_0(kR)I_0'(kR)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. + \sigma_1 I_0(kR)K_0'(kR)\right) + (\sigma_1 - \sigma_2)^2\left(K_0(kR)I_0(kR) + K_0'(kR)I_0'(kR)\right)^2\right)/D_1\right\}\right\}, \\ D_1 &= \sigma_1 I_0'(kR)K_0(kR) - \sigma_2 I_0(kR)K_0'(kR).\end{aligned}$$

References

[1] P.C. Duineveld, The stability of ink-jet printed lines of liquid with zero receding contact angle on a homogeneous substrate, J. Fluid Mech. 477 (2003) 175.
[2] J.M. Crowley, The onset of boiling in electrohydrodynamic spraying, AIAA 15 (1977) 734.
[3] A.D. Moore, Electrostatics and its Applications, John Wiley & Sons, New York, 1973.
[4] J. Plateau, Experimental and theoretical researches on the figures of equilibrium of a liquid mass withdrawn from the action of gravity, etc., Annual Report of the Board of Regents of the Smithsonian Institution, Pt. 6, House of representatives Misc. Doc. # 83, 39th Congress, 2nd Session, pp. 255–289 (1863–1866).
[5] L. Rayleigh, On the instability of jets, Proc. London Math. Soc. 10 (1879) 4.
[6] C. Weber, Zum Zerfall eines Flüssigkeitsstrahles, Z. Angew. Math. Mech. 11 (1931) 136.
[7] S. Tomotika, On the instability of a cylindrical thread of a viscous liquid surrounded by another viscous fluid, Proc. R. Soc. Lond. A 150 (1935) 322.
[8] L. Rayleigh, On the instability of a cylinder of viscous liquid under capillary force, Philos. Mag. 34 (1892) 145.
[9] W.K. Lee, R.W. Flumerfelt, Instability of stationary and uniformly moving cylindrical fluid bodies – I. Newtonian systems, Int. J. Multiphase Flow 7 (1981) 363.
[10] T. Funada, D.D. Joseph, Viscous potential flow analysis of capillary instability, Int. J. Multiphase Flow 28 (2002) 1459.
[11] T. Funada, D.D. Joseph, Viscoelastic potential flow analysis of capillary instability, J. Non-Newtonian Fluids 11 (2002) 87.
[12] D.D. Joseph, G.S. Beavers, T. Funada, Rayleigh–Taylor instability of viscoelastic drops at high Weber numbers, J. Fluid Mech. 453 (2002) 109.
[13] G.I. Taylor, Disintegration of water drops in an electric field, Proc. R. Soc. Lond. A 280 (1964) 383.
[14] G.I. Taylor, Studies in electrohydrodynamics. I. The circulation produced in a drop by an electric field, Proc. R. Soc. Lond. A 291 (1966) 159.
[15] G.I. Taylor, Electrically driven jet, Proc. R. Soc. Lond. A 313 (1969) 453.
[16] J.R. Melcher, G.I. Taylor, Electrohydrodynamics: A review of the role of interfacial shear stresses, Annu. Rev. Fluid Mech. 1 (1969) 111.
[17] D.A. Saville, Electrohydrodynamic stability: Fluid cylinders in longitudinal electric fields, Phys. Fluids 13 (1970) 2987.
[18] D.A. Saville, Electrohydrodynamics: The Taylor–Melcher leaky dielectric model, Annu. Rev. Fluid Mech. 29 (1997) 27.
[19] A.S. Mestel, Electrohydrodynamic stability of a slightly viscous jet, J. Fluid Mech. 274 (1994) 93.
[20] A.S. Mestel, Electrohydrodynamic stability of a highly viscous jet, J. Fluid Mech. 312 (1996) 311.
[21] J.Q. Feng, T.C. Scott, A computational analysis of electrohydrodynamics of a leaky dielectric drop in an electric field, J. Fluid Mech. 311 (1996) 289.
[22] J.Q. Feng, Electrohydrodynamic behaviour of a drop subjected to a steady uniform electric field at finite electric Reynolds number, Proc. R. Soc. Lond. A. 455 (1999) 2245.
[23] A.R.F. Elhefnawy, B.M.H. Agoor, A.E.K. Elcoot, Nonlinear electrohydrodynamic stability of a finitely conducting jet under an axial electric field, Physica A 297 (2001) 368.
[24] A.R.F. Elhefnawy, G.M. Moatimid, A.E.K. Elcoot, Nonlinear streaming instability of cylindrical structures in finitely conducting fluids under the influence of a radial electric field, Physica Scripta 67 (2003) 513.
[25] A.R.F. Elhefnawy, G.M. Moatimid, A.E.K. Elcoot, The effect of an axial electric field on the nonlinear stability between two uniform stream flows of finitely conducting cylinders, Can. J. Phys. 81 (2003) 805.
[26] A.R.F. Elhefnawy, G.M. Moatimid, A.E.K. Elcoot, Nonlinear electrohydrodynamic instability of a finitely conducting cylinder. Effect of interfacial surface charges, Z. Angew. Math. Mech. 55 (2004) 63.
[27] A.E.K. Elcoot, G.M. Moatimid, Nonlinear instability of finitely conducting cylindrical flows through porous media, Physica A 343 (2004) 15.
[28] M.F. El-Sayed, A.A. Mohamed, T.M.N. Metwaly, Thermohydrodynamic instabilities of conducting liquid jets in the presence of time-dependent transverse electric fields, Physica A 345 (2005) 367.
[29] M.F. El-Sayed, A.A. Mohamed, T.M.N. Metwaly, Effect of general applied electric field on conducting liquid jets instabilities in the presence of heat and mass transfer, Appl. Math. Comput. 172 (2006) 1078.
[30] J.Q. Feng, K.V. Beard, Resonances of a conducting drop in an alternating electric field, J. Fluid Mech. 222 (1991) 417.
[31] T. Funada, D.D. Joseph, Viscous potential flow analysis of Kelvin–Helmholtz instability in a channel, J. Fluid. Mech. 445 (2001) 263.
[32] C.L. Burcham, D.A. Saville, The electrohydrodynamic stability of a liquid bridge: microgravity experiments on a bridge suspended in a dielectric gas, J. Fluid Mech. 405 (2000) 37.
[33] C.L. Burcham, D.A. Saville, Electrohydrodynamic stability: Taylor–Melcher theory for a liquid bridge suspended in a dielectric gas, J. Fluid Mech. 452 (2002) 163.
[34] R. Grimshaw, Modulation of an internal gravity wave packet in a stratified shear flow, Wave Motion 3 (1981) 81.
[35] J.Q. Feng, Electrohydrodynamic model for electrorotation of fluid drops, J. Colloid Interface Sci. 246 (2002) 112.
[36] Y.O. El-Dib, Nonlinear Rayleigh–Taylor instability for hydromagnetic Darcian flow: effect of free surface currents, J. Colloid Interface Sci. 309 (2003) 321.
[37] G.M. Moatimid, Non-linear electrorheological instability of two streaming cylindrical fluids, J. Phys. A Math. Gen. 36 (2003) 11343.
[38] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford Univ. Press, London, 1961.
[39] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1973.
[40] C.G. Lange, A.C. Newell, A stability criterion for envelope equations, SIAM J. Appl. Math. 27 (1974) 441.
[41] A.H. Nayfeh, Nonlinear propagation of wave packets on fluid-interfaces, Trans. ASME 98E (1976) 584.
[42] T. Iizuka, M. Wadati, The Rayleigh–Taylor instability and nonlinear waves in a two-dimensional jet and wake, Fluid Dynam. Res. 4 (1990) 15.
[43] A.R.F. Elhefnawy, The nonlinear stability of mass and heat transfer in magnetic fluids, Z. Angew. Math. Mech. 77 (1997) 19.